Economic Theory II
Lecture 4 [1]

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[1] Content of current lecture slides derives on slides prepared by Dr Valentin Jouvanceau previously.

Issues with the Solow Model [2]

- We saw how the saving rate can determine if consumption is at the maximum possible or not.
- Solow model does not consider how the saving decision is arrived at, as preferences and decision making are not modelled.
- This setting also makes policy analysis impossible, as optimal response from decision makers not modelled.
- Also makes it difficult to model intergenerational transfers due to infinitely lived household setting.
- Solution? The overlapping Generations (OLG) model by Paul Samuelson and Peter Diamond (1965).

[2] Chapter 8 of GLS.

OLG Model

Households

- At any given time period t, there are two types of agents (or, equivalently, households) alive: 'young' and 'old'. The 'old' in t were 'young' in $t-1$, and the 'young' in t are born in t.
- Each agent lives two periods; it is 'young' in it's first period of life, and 'old' in it's second period of life.
- If N_t is the number of 'young' born in each period t, the total number of people (young + old) in any given t is $N_t + N_{t-1}$.
- Assume that $N_t > N_{t-1} \ \ \forall t \in \{1, 2, ...\}$.

OLG Model

- A household born in period t has the budget constraint:

$$c_{y,t} + s_t \leq w_t \tag{1}$$

where $c_{y,t}$ is consumption of young born in t, s_t is saving, which is turned into productive capital, and w_t are the wages.

- Same household, when old, has the following budget constraint:

$$c_{o,t+1} \leq [R_{t+1} + (1 - \delta)]s_t \tag{2}$$

where $c_{o,t+1}$ is the consumption in $t + 1$ (when old) of the household which was born in t, $R_{t+1}s_t$ is the total rent received from productive capital, and $(1 - \delta)s_t$ is the value of undepreciated capital. (2) says that you cannot consume more than what you have.

OLG Model

- Lifetime utility to be maximised is :

$$U = u(c_{y,t}) + \beta u(c_{o,t+1}) \tag{3}$$

and the constraints that determine how much one can consume each period are (1) and (2).

- Maximise the utility by choosing how much to save, so that it implies how much you consume today as well as tomorrow:

$$\max_{s_t} U = u(w_t - s_t) + \beta u([R_{t+1} + (1-\delta)]s_t) \tag{4}$$

- Optimising condition:

$$\frac{\partial U}{\partial s_t} = 0 \Leftrightarrow u'(w_t - s_t) = \beta[R_{t+1} + (1-\delta)]u'([R_{t+1} + (1-\delta)]s_t) \tag{5}$$

OLG Model

- Implicitly, from (5), saving is a function of wages w_t and the rental rate R_{t+1} : $s_t = s(w_t, R_{t+1})$.
- Without function forms for utility etc., we can only say that as $w_t \uparrow\downarrow$, $s_t \uparrow\downarrow$ in the same direction, so that (5) will always hold.
- However, impact of R_{t+1} ambiguous. Depends on the functional forms, parameter values, etc. (Why?)

OLG Model

Production

- Output is produced using aggregate capital K_t, which is the total saving of all households, and labour N_t which is supplied only by the young each period:

$$Y_t = AF(K_t, N_t) \tag{6}$$

- Profit maximisation is as under:

$$\max_{K_t, N_t} \Pi_t = AF(K_t, N_t) - w_t N_t - R_t K_t \tag{7}$$

- Factor prices resulting from optimality are:

$$AF_K(K_t, N_t) = R_t$$
$$AF_N(K_t, N_t) = w_t$$

OLG Model

Equilibrium

- Aggregate saving of all households each period become capital for next period: $K_{t+1} = N_t s_t$.
- Aggregate consumption of young each period:

$$N_t c_{y,t} + N_t s_t = N_t w_t \tag{8}$$

- Aggregate consumption of old each period:

$$N_{t-1} c_{o,t} = [R_t + (1 - \delta)] N_{t-1} s_{t-1} \tag{9}$$

- Let $C_{y,t} = N_t c_{y,t}$ and $C_{o,t} = N_{t-1} c_{o,t}$.
- Using that in (8) and 9:

$$C_{y,t} + K_{t+1} = N_t w_t \tag{10}$$

$$C_{o,t} = [R_t + (1 - \delta)] K_t \tag{11}$$

OLG Model

- Summing (10) and (11):

$$C_{y,t} + C_{o,t} + K_{t+1} - (1-\delta)K_t = w_t N_t + R_t K_t \tag{12}$$

- Using Constant Returns to Scale property of the production function (6), $N_t w_t + R_t K_t = Y_t$, and:

$$C_t + I_t = Y_t \tag{13}$$

where total consumption of young and old in t is $C_t = C_{y,t} + C_{o,t}$.

OLG Model Equilibrium

Full system:

$$
\begin{aligned}
s_t &= s(w_t, R_{t+1}) \\
K_{t+1} &= N_t s_t \\
I_t &= K_{t+1} - (1-\delta)K_t \\
Y_t &= C_t + I_t \\
Y_t &= AF(K_t, N_t) \\
R_t &= AF_K(K_t, N_t) \\
w_t &= AF_N(K_t, N_t) \\
C_t &= C_{y,t} + C_{o,t} \\
C_{y,t} &= N_t w_t - K_{t+1} \\
N_t &= (1+n)N_{t-1}
\end{aligned}
$$

9 equations in 9 variables (except n and A which are exogenous):

$$s,\ w,\ R,\ K,\ I,\ Y,\ C,\ C_y,\ C_o$$

OLG Model Equilibrium

- We convert to *per capita* terms by dividing by N_t, N_{t+1}, or N_{t-1}, as appropriate.

- Some important transformations:
  - $\frac{K_{t+1}}{N_{t+1}} = \frac{N_t}{N_{t+1}} s_t = \frac{s_t}{1+n}$
  - $\frac{C_t}{N_t} = \frac{C_{y,t}}{N_t} + \frac{C_{o,t}}{N_t} \Rightarrow c_t = c_{y,t} + \frac{N_{t-1}}{N_{t-1}} \frac{C_{o,t}}{N_t} \Rightarrow c_t = c_{y,t} + \frac{c_{o,t}}{1+n}$
  - $\frac{I_t}{N_t} = \frac{N_{t+1}}{N_{t+1}} \frac{K_{t+1}}{N_t} - (1-\delta)\frac{K_t}{N_t} \Rightarrow i_t = (1+n)k_{t+1} - (1-\delta)k_t$
  - $\frac{C_{y,t}}{N_t} = \frac{N_t}{N_t} w_t - \frac{N_{t+1}}{N_{t+1}} \frac{K_{t+1}}{N_t} \Rightarrow c_{y,t} = w_t - (1+n)k_{t+1}$

OLG Model Equilibrium

Full per capita system:

$$s_t = s(w_t, R_{t+1}) \tag{14}$$

$$k_{t+1} = \frac{s_t}{1+n} \tag{15}$$

$$i_t = (1+n)k_{t+1} - (1-\delta)k_t \tag{16}$$

$$y_t = c_t + i_t \tag{17}$$

$$y_t = Af(k_t) \tag{18}$$

$$R_t = Af'(k_t) \tag{19}$$

$$w_t = Af(k_t) - Af'(k_t)k_t \tag{20}$$

$$c_t = c_{y,t} + \frac{c_{o,t}}{1+n} \tag{21}$$

$$c_{y,t} = w_t - (1+n)k_{t+1} \tag{22}$$

Most important equation, combining (14), (19), and (20):

$$k_{t+1} = \frac{s(Af(k_t) - Af'(k_t)k_t, Af'(k_{t+1}))}{1+n} \tag{23}$$

OLG Functional Forms

- Let utility be logarithmic: $u(.) = ln(.)$. Then, (5) becomes:

$$\frac{1}{w_t - s_t} = \beta[R_{t+1} + (1-\delta)]\frac{1}{[R_{t+1} + (1-\delta)]s_t}$$
$$\Rightarrow s_t = \underbrace{\frac{\beta}{1+\beta}}_{<1} w_t \tag{24}$$

- No impact of R_{t+1} on s_t in this functional form!

OLG Functional Forms

- Let production be Cobb-Douglas: $F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$. Then, wages become:

$$\begin{aligned} w_t &= (1-\alpha) A K_t^{\alpha} N_t^{-\alpha} \\ \Rightarrow w_t &= (1-\alpha) A k_t^{\alpha} \end{aligned} \tag{25}$$

- Using (25) and (24) in (23):

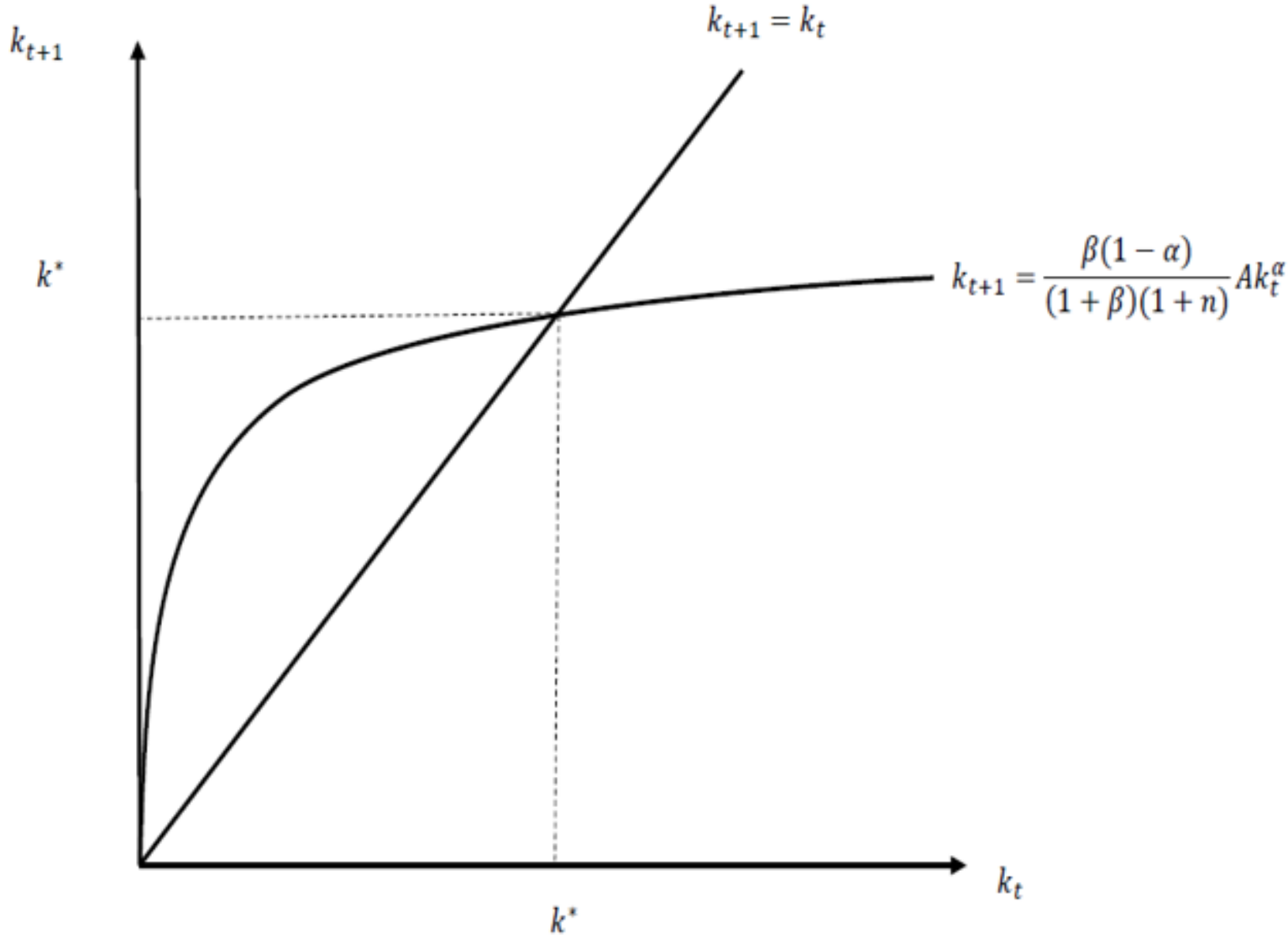
$$k_{t+1} = \frac{\frac{\beta}{1+\beta}(1-\alpha) A k_t^{\alpha}}{1+n} \tag{26}$$

- More cleanly:

$$k_{t+1} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} k_t^{\alpha} \tag{27}$$

OLG Functional Forms

Plotting (27):



OLG vs. Solow

- Solving (27) in the steady state for k^*:

$$k^* = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)} \right)^{\frac{1}{1-\alpha}} \tag{28}$$

- Comparing to the steady state expression for Solow model with only population growth:

$$k^* = \left(\frac{sA}{\delta + n} \right)^{\frac{1}{1-\alpha}} \tag{29}$$

- Primary difference: k^* in OLG does not depend on depreciation rate δ! Why?

- Think about main difference between the two models. Does that provide an explanation?

- Does time have the same interpretation in both models?

OLG and The Golden Rule

- In steady state, investment is : $i^* = (\delta + n)k^*$.
- Steady state consumption is:
  $c^* = y^* - (\delta + n)k^* = Af(k^*) - (\delta + n)k^*$.
- For the Golden Rule, we maximise consumption by choosing k^*. So;

$$\frac{dc^*}{dk^*} = 0 \Leftrightarrow Af'(k^*) - (\delta + n) = 0 \Leftrightarrow Af'(k^*) = \delta + n \tag{30}$$

- In the Cobb-Douglas form:

$$\alpha A(k^*)^{\alpha - 1} = \delta + n$$

$$k^*_{gr} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}} \tag{31}$$

OLG and The Golden Rule

- What conditions are necessary for the steady state capital to be the Golden Rule?

$$k^* = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+n)}\right) = \left(\frac{\alpha A}{\delta+n}\right) = k^*_{gr}$$
$$\Rightarrow \delta + n = \left(\frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)}\right) \tag{32}$$

- If δ very high, for instance, then, $k^* > k^*_{gr}$ possibly. This is means economy is *dynamically inefficient*: increase consumption and reduce saving to reduce k^* and get to k^*_{gr}.

OLG and Policy

- But a very high δ causes return to savings $[R + (1 - \delta)]$ to be low as capital depreciates much faster.
- This is important because savings is the only means of providing for future consumption by young people.
- Can there be some 'policy' that can lead to a better outcome?
- What if government taxes savings of young agents and subsidises consumption of old agents?
- Population is growing by assumption $n > 0$, so this policy is sustainable.

OLG and Policy

- Take τ_s proportion of savings from young in t:

$$c_{y,t} + (1 + \tau_s)s_t \leq w_t$$

.

- When that household is old, give it additional capital income τ_k:

$$c_{o,t+1} \leq [(1 + \tau_k)R_{t+1} + (1 - \delta)]s_t$$

.

- The government balances it's budget:

$$\tau_s N_t s_t = \tau_k R_t N_{t-1} s_{t-1}$$

.

- The optimality condition, with log utility, becomes:

$$\begin{aligned} \frac{1 + \tau_s}{w_t - (1 + \tau_s)s_t} &= \beta \frac{1}{s_t} \\ \Rightarrow s_t &= \frac{\beta}{(1 + \beta)(1 + \tau_s)} \underbrace{(1 - \alpha)Ak_t^{\alpha}}_{w_t} \end{aligned} \tag{33}$$

OLG and Policy

- The equation for capital becomes:

$$k_{t+1} = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+\tau_s)(1+n)} \right) k_t^{\alpha} \tag{34}$$

- In steady state:

$$k^* = \left(\frac{\beta(1-\alpha)A}{(1+\beta)(1+\tau_s)(1+n)} \right)^{\frac{1}{1-\alpha}} \tag{35}$$

- For the above to be the Golden Rule capital stock;

$$\frac{\beta(1-\alpha)}{(1+\beta)(1+\tau_s)(1+n)} = \frac{\alpha}{\delta+n}$$

$$\tau_s = \frac{\beta(1-\alpha)(\delta+n) - (1+\beta)\alpha(1+n)}{\alpha(1+\beta)(1+n)} \tag{36}$$

OLG and Policy

- When is $\tau_s > 0$?

$$\begin{aligned} \beta(1-\alpha)(\delta+n) &> (1+\beta)\alpha(1+n) \\ \Rightarrow \delta + n &> \frac{(1+\beta)\alpha(1+n)}{\beta(1-\alpha)} \end{aligned} \tag{37}$$

- Condition for dynamic inefficiency! Impose tax when economy dynamically inefficient.
- What if economy is *dynamically efficient*?
- Condition for balanced budget:

$$\tau_k = \frac{(1-\alpha)\beta\tau_s}{\alpha(1+\beta)(1+\tau_s)} \tag{38}$$

- Check quantitative experiments in the GLS book (pages 168-172) to understand Pareto Efficiency.

End